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LETTER TO THE EDITOR

The three-spin Ising model as an eight-vertex model

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Abstract. The three-spin Ising model is related to a special case of the eight-vertex model by a spin transformation. It follows that the models have the same free energy, magnetization and polarization.

The solution of the zero-field eight-vertex model (Baxter 1971, 1972) includes as special cases most of the known exact solutions of two-dimensional lattice models. One model not included is the three-spin Ising model (or triplet model) which can be defined as having the energy given by (Wood and Griffiths 1972)

$$E=\sum_{ij} E_{ij}^{(T)}=-\sum_{ij} J\sigma_{ij}\sigma_{i+1,j+1}(\sigma_{i,j+1}+\sigma_{i+1,j}). \tag{1}$$

The sites (i, j) are on a square lattice and the σ_{ij} are Ising spins taking values ± 1. The free energy of this model has also been obtained exactly (Baxter and Wu 1974) and the model does not at first appear to be equivalent to any of the eight-vertex models. There are however a number of points of similarity between the solutions. Both can be naturally described in terms of elliptic functions which lead to expressions involving infinite sums and products as discussed by Baxter *et al* (1975).

In particular, for temperatures T less than the critical temperature T_c, if p is defined by

$$\exp(-4\beta J)=u=p\prod_{n=1}^{\infty}\left(\frac{(1-p^{8n-7})(1-p^{8n-1})}{(1-p^{8n-5})(1-p^{8n-3})}\right)^2, \qquad 0<p<1, \tag{2}$$

then the triplet model free energy is given by

$$-\beta f=2\beta J+\ln\prod_{n=1}^{\infty}\frac{(1-p^{6n-3})(1-p^{8n-4})^3(1-p^{8n})}{(1-p^{6n})(1-p^{8n-5})^2(1-p^{8n-3})^2} \tag{3}$$

or equivalently (expanding the logarithms and summing over n)

$$-\beta f=2\beta J+\sum_{j=1}^{\infty}\frac{p^{3j}(1-p^j)^4(1+p^j)}{j(1-p^{8j})(1+p^{3j})}. \tag{4}$$

It has been conjectured that the magnetization and polarization are given by

$$M=\prod_{n=1}^{\infty}\frac{(1-p^{6n-3})}{(1+p^{6n-3})} \tag{5}$$

and

$$P=\prod_{n=1}^{\infty}\left(\frac{(1-p^{3n})(1+p^{4n})}{(1+p^{3n})(1-p^{4n})}\right)^2. \tag{6}$$

These expressions are remarkably similar to those of the eight-vertex model. Comparison of (5) with equation (15) of Barber and Baxter (1973), (6) with equation (18) of Baxter and Kelland (1974) and (4) with equation (D37) of Baxter (1972) shows that the expressions (4), (5), (6) are those of an eight-vertex model with

$$x=p^{3/2} \tag{7}$$

$$q=p^4 \tag{8}$$

$$z=p^{-1/2}. \tag{9}$$

The elliptic function parameters x, q, z are defined by equation (D8) of Baxter (1972). We find that these parameters correspond to an eight-vertex model with energy

$$E=\sum_{ij}E_{ij}^{(8)}=-\sum_{ij}J\sigma_{ij}\sigma_{i+1,j+1}(1+\sigma_{i,j+1}\sigma_{i+1,j}). \tag{10}$$

This is a square lattice model with a two-spin interaction only along one diagonal direction. The interaction energy J is related to the parameter p by equation (2), i.e. it has the same value as in (1).

One can also verify that the two models satisfy the same duality relation, and so have the same free energy for $T>T_c$.

The agreement between the solutions suggests the existence of some kind of transformation of the three-spin model into an eight-vertex model. For the case of a rectangular lattice with all boundary spins fixed at $+1$ we are able to build up one such transformation by the following iterative procedure.

The transformation starts from an invariant line such as the heavy diagonal zig-zag line of figure 1. From this line we build up a central band of transformed squares so that at each stage of the interaction the energy, as a function of the transformed spins, is given by

$$E=\sum_{ij}^{(1)}E_{ij}^{(8)}+\sum_{ij}^{(2)}E_{ij}^{(T)} \tag{11}$$

where $\Sigma^{(1)}$ is over all squares in the central band and $\Sigma^{(2)}$ is over all other squares. To add an additional line of transformed squares we perform the following transformation σ to σ^*.

(i) All spins in and below the central band are fixed, so that $\sigma_{ij}^*=\sigma_{ij}$.

(ii) The outermost spins on the upper boundary of the central band are all on one of the three triangular sublattices described by Baxter and Wu (1974). This sublattice $\mathscr{L}_1$ is shown by open circles in figure 1. The spins on these sites are left unchanged ($\sigma_{ij}^*=\sigma_{ij}$) but they serve to define the transformation of those on the other sites.

(iii) Each other spin σ_{ij} above the central band has three neighbours σ_x, σ_y, σ_z on the invariant sublattice $\mathscr{L}_1$ of (ii). The transformation of σ_{ij} is

$$\sigma_{ij}^*=\sigma_{ij}\times\mathrm{sgn}(\sigma_x+\sigma_y+\sigma_z). \tag{12}$$

This transformation can be considered to apply to the boundary sites if the lattice is treated as though it were completely surrounded by further sites fixed at $+1$. The consequence of this is that the true fixed boundary sites are always left unchanged at $+1$.

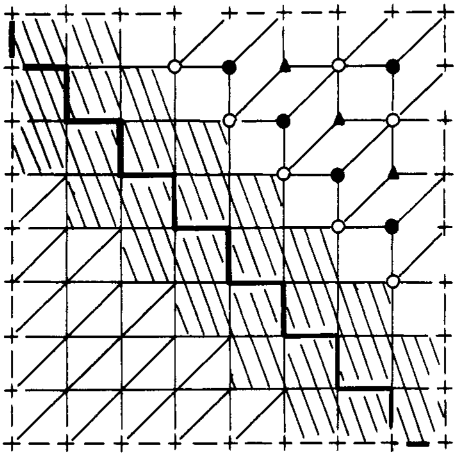

Figure 1. Transformation of the three-spin model into an eight-vertex model. The central band shaded has its energy given by the eight-vertex model expression and contains the invariant starting line (the heavy diagonal line). Outside the central band the energy is a three-spin interaction summed over triangles as shown in the lower left corner. The upper right corner shows the combination of pairs of triangles into dice lattice faces and also shows the three-sublattice division. The invariant sublattice $\mathscr{L}_1$ is shown by open circles. The fixed boundary spins are denoted +.

Obviously the transformation will change the energy associated with many of the elementary triangles but if the triangles are combined in pairs to form the faces of a dice lattice as shown in figure 1 then it is found that the transformation leaves unchanged the energy of any face not having an edge lying on the boundary of the central band.

To see this, consider a quadrilateral as shown in figure 2. In terms of the original variables the energy is given by $E=-J(\sigma_a+\sigma_c)\sigma_b\sigma_d$. Since a and c are on the invariant sublattice we have $\sigma_a^*=\sigma_a$, $\sigma_c^*=\sigma_c$. There are three cases to consider which are outlined below.

(i) $\sigma_a=\sigma_c=+1$. Both b and d have at least two positive neighbour spins in $\mathscr{L}_1$, so the sign in (12) is positive and $\sigma_b^*=\sigma_b$, $\sigma_d^*=\sigma_d$ and $E=-J(\sigma_a^*+\sigma_c^*)\sigma_b^*\sigma_d^*$.

(ii) $\sigma_a=\sigma_c=-1$. Thus $\sigma_b^*=-\sigma_b$, $\sigma_d^*=-\sigma_d$ and $E=-J(\sigma_a^*+\sigma_c^*)\sigma_b^*\sigma_d^*$.

(iii) $\sigma_a=-\sigma_c$, $E=0$. Now σ_b^* and σ_d^* depend on spins outside the face *abcd*. However, as for cases (i) and (ii), it remains true that $(\sigma_a+\sigma_c)\sigma_b\sigma_d=(\sigma_a^*+\sigma_c^*)\sigma_b^*\sigma_d^*$ since both sides are zero independently of σ_b^*, σ_d^*.

An exception to this argument occurs at the boundaries where isolated triangles can occur due to the misfit between the dice lattice and the square lattice. These triangles all

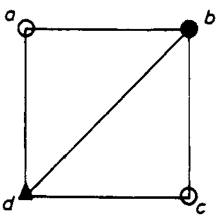

Figure 2. A typical dice lattice face showing the indexing used in investigating the energy transformation.

have two fixed boundary sites and one site on the invariant sublattice so that the energy of any such triangle is left unchanged.

For squares sharing an edge with the central band, σ_d (or σ_b for squares below the central band) is unchanged: $\sigma_d^* = \sigma_d$. Again $\sigma_a^* = \sigma_a$, $\sigma_c^* = \sigma_c$ and the same three cases must be considered.

We find that $E = -J(\sigma_a + \sigma_c)\sigma_b\sigma_d = -J\sigma_b^*\sigma_d^*(1 + \sigma_a^*\sigma_c^*)$ for each case:

(i) $\sigma_a = \sigma_c = +1$, $\sigma_b^* = \sigma_b$ and $E = -2J\sigma_b\sigma_d = -2J\sigma_b^*\sigma_d^*$;

(ii) $\sigma_a = -\sigma_c$ and $E = 0$ independently of σ_b^*;

(iii) $\sigma_a = \sigma_c = -1$, $\sigma_b^* = -\sigma_b$ and $E = 2J\sigma_b\sigma_d = -2J\sigma_b^*\sigma_d^*$.

Thus the transformation changes the interaction Hamiltonian for these squares from that of the three-spin model (equation (1)) to that of the eight-vertex model (equation (10)). Hence one diagonal line of squares is added to the central band of figure 1.

The transformation can be applied equally well to the region below the central band and after sufficient iterations any finite lattice with rectangular boundaries and fixed boundary sites (as in figure 1) can be converted from a three-spin model to an eight-vertex model. The equivalence of the free energies then follows by taking the large size limit. The magnetization and polarization can be defined by $M = \langle\sigma_{ij}\rangle$, $P = \langle\sigma_{ij}\sigma_{i,j+1}\rangle$ for sites sufficiently far from the boundary. Sites on the invariant starting line can be used to define these order parameters which must therefore be the same for each model. There will be a large class of diagonal correlations (namely those only involving spins on the invariant line) that are the same for both models.

Since the expressions for the magnetization and polarization were conjectured on different grounds for each model (Barber and Baxter 1973, Baxter and Kelland 1974, Baxter *et al* 1975) the equivalence derived here is further evidence in support of these conjectures.

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